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CARATHÉODORY BALLS AND NORM BALLS IN $H_n = \{z \in \mathbb{C}^n : \|z\|_1 < 1\}$

BY

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In memory of Albert Pfluger

ABSTRACT

It is shown that the only balls in the Carathéodory distance on $H_n = \{z \in \mathbb{C}^n : \|z\|_1 < 1\}$, $n \geq 2$, which are balls with respect to the ℓ_1 norm in $\mathbb{C}^n$ are those centered at the origin.

1. Introduction

Consider the unit ball

$$H = H_n = \{z \in \mathbb{C}^n : \|z\|_1 < 1\}$$

in the complex n-space $\mathbb{C}^n$ with respect to the metric which is induced by the ℓ_1 norm in $\mathbb{C}^n$

$$\|z\| = \|z\|_1 = \sum_{k=1}^{n} |z_k|, \qquad z = (z_1, \ldots, z_n) \in \mathbb{C}^n.$$

Next consider the Carathéodory distance $C = C_H$ on H

$$C(z, w) = \sup \rho(f(z), f(w)), \qquad z, w \in H,$$

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where the supremum is taken over all holomorphic functions f from H into the unit disk

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}$$

of the complex plane $\mathbb{C}$. Here ρ is the hyperbolic distance on Δ:

$$\rho(a,b) = \frac{1}{2}\log\frac{1+\alpha}{1-\alpha} = \tanh^{-1}\alpha, \quad \text{where} \quad \alpha = \left|\frac{a-b}{1-a\bar{b}}\right|, \quad a,b \in \Delta.$$

Note that on H_n, the Carathéodory distance and the Kobayashi distance are the same.

For $n = 1$, $H_1 = \Delta$, and by Schwarz–Pick Theorem, $C(a,b) = \rho(a,b)$, $a,b \in \Delta$, and $\|z\| = |z|$, $z \in \Delta$, and since, cf. [4, Lemma 2.1],

$$\rho(z,a) = R \Leftrightarrow |z-b| = r, \quad a,b,z \in \Delta, \tag{1.1}$$

where

$$b = a\frac{1-\alpha^2}{1-\alpha^2|a|^2} \quad \text{and} \quad r = \alpha\frac{1-|a|^2}{1-\alpha^2|a|^2}, \quad \alpha = \tanh R, \tag{1.2}$$

it follows that in H_1 every ball (i.e. disk) in the Carathéodory distance on H_1 is a ball with respect to the ℓ_1 norm in $\mathbb{C}$.

A general theorem on Carathéodory distance on normed spaces, cf. [2, Theorem IV 1.8] implies that every ball in the Carathéodory distance on H_n, $n \geq 1$, which is centered at the origin is a ball in the ℓ_1 norm of $\mathbb{C}^n$. The following result of Binyamin Schwarz shows that in $\mathbb{C}^2$ Carathéodory balls of H_2 which are centered off the origin are not balls in the ℓ_1 norm of $\mathbb{C}^2$.

THEOREM A ([4, Theorem 3.1]): *The only balls in $H_2 = \{z \in \mathbb{C}^2 : \|z\| < 1\}$ in the Carathéodory distance on H_2 which are balls in the ℓ_1 norm in $\mathbb{C}^2$ are those which are centered at the origin.*

The following geometric lemma which Schwarz establishes in [4] is repeatedly applied in the proof of Theorem A.

LEMMA ([4, Lemma 2.4]): *Let γ_1 and γ_2 be circles in the complex plane $\mathbb{C}$,*

$$\gamma_k = \{z : z = z_k(\varphi) = a_k + r_k e^{i(\varphi+\nu_k)}, r_k > 0, -\infty < \varphi < \infty\}, \quad k = 1,2.$$

Then, a necessary and sufficient condition for the existence of two points ζ_1 and ζ_2 in $\mathbb{C}$ and real numbers ν_1 and ν_2 such that the equality

$$\sum_{k=1}^{2} |z_k(\varphi) - \zeta_k| = \text{constant} \qquad \text{for } -\infty < \varphi < \infty$$

holds, is $\zeta_k = a_k, k = 1, 2$.

It is conjectured in [4] that the statement stated in the lemma holds for any number $n \geq 2$ of circles γ_k and points ζ_k, and it is pointed out that the generalization of the lemma for $n \geq 2$ would imply a generalization of the theorem for H_n, $n \geq 2$, with a proof virtually the same as the one brought in [4] for H_2.

The purpose of this note is to show that, indeed, the lemma and the theorem hold for any number $n \geq 2$. An extension of the lemma is contained in the following proposition.

The novelty of this note is in the proof of the proposition. The proof of the extension of Theorem A for $n \geq 2$ is the same as in [4], though set up differently.

2. A proposition

PROPOSITION: *Given*

(i) *n circles $\gamma_k = \{z \in \mathbb{C} : |z - a_k| = r_k > 0\}$, $k = 1, \dots, n$,*

(ii) *n points $z_1, \dots, z_n$ in motion such that the point z_k moves along γ_k with state equation*

$$z_k(t) = a_k + r_k e^{i(t+\nu_k)}, \quad -\infty < t < \infty, \quad k = 1, \dots, n,$$

where the phases $\nu_1, \dots, \nu_n$ are given,

(iii) *n points $b_1, \dots, b_n$ in $\mathbb{C}$,*

(iv) *n real positive numbers $\lambda_1, \dots, \lambda_n$ and a real positive number c such that*

$$\sum_{k=1}^{n} \lambda_k |z_k(t) - b_k| \equiv c, \quad -\infty < t < \infty, \tag{2.1}$$

then $a_k = b_k$ for all $k = 1, \dots, n$.

Proof of the Proposition: If for some $k, a_k = b_k$ as desired, then the term $|z_k(t) - b_k|$ yields a constant contribution to the sum in (2.1), and may be dropped. We thus may assume that $a_k \neq b_k$ for all $k = 1, \dots, n$. By rotating, translating, rescaling and renaming the constants c, λ_k, b_k and ν_k we may assume that for all $k = 1, \dots, n$, $a_k = 0$, $r_k = 1$ and b_k is real and positive. Then $|z_k(t)| = 1$, $\arg z_k(t) = t + \nu_k$, and by the Cosine Theorem, (2.1) becomes

$$\sum_{k=1}^{n} \lambda_k \left(1 + b_k^2 - 2b_k \cos(t + \nu_k)\right)^{1/2} \equiv c, \quad -\infty < t < \infty \tag{2.2}$$

or

$$\sum_{k=1}^{n} f_k(t) \equiv c, \quad -\infty < t < \infty, \tag{2.3}$$

where

$$f_k(t) = \lambda'_k\big(A_k - \cos(t+\nu_k)\big)^{1/2}, \quad k = 1,\ldots,n, \tag{2.4}$$

and where $\lambda'_k = \lambda_k(2b_k)^{1/2}$ and $A_k = \frac{1}{2}(b_k + b_k^{-1})$. As $b_k > 0, A_k \geq 1$.

Now let

$$\varphi_k(z) = A_k - \cos(z+\nu_k), \quad z \in \mathbb{C}, \quad k = 1,\ldots,n, \quad \text{and} \tag{2.5}$$

$$Z_k = \{z \in \mathbb{C} : \varphi_k(z) = 0\}, \quad k = 1,\ldots,n. \tag{2.6}$$

Then $A_k = 1 \Rightarrow Z_k = \{w_k + 2m\pi : m \in Z\}$ for some real number w_k, and $A_k > 1 \Rightarrow Z_k = \{w_k + 2m\pi : m \in Z\} \cup \{\bar{w}_k + 2m\pi : m \in Z\}$ for some non-real number w_k in $\mathbb{C}$.

We may assume that $\varphi_j \neq \varphi_k$ for all $j \neq k$, since otherwise the corresponding terms in (2.3) may be grouped together. Then $A_j \neq A_k$ or $\nu_j \neq \nu_k (\bmod 2\pi)$ for $j \neq k$. In the first case $|\mathrm{Im}\, w_j| \neq |\mathrm{Im}\, w_k|$, and in the latter case $\mathrm{Re}\, w_j \neq \mathrm{Re}\, w_k (\bmod 2\pi)$ for any $w_j \in Z_j$ and $w_k \in Z_k$ with $j \neq k$. Therefore

$$Z_j \cap Z_k = \phi \quad \text{for all} \quad j \neq k. \tag{2.7}$$

Suppose now that $A_j = 1$. Then $\varphi_j(-\nu_j) = 0$ and $\varphi_k(-\nu_j) \neq 0$ for all $k \neq j$. Then $f_k(t) = \lambda'_k\varphi_k(t)^{1/2}$ is real analytic at $t = -\nu_j$, for all k, $k \neq j$, and by (2.3) so is $\sum_{k=1}^{n} f_k(t)$. Hence, also $f_j(t)$ must be real analytic at $t = -\nu_j$. But f_j is not real analytic (it is not even differentiable) at $t = -\nu_j$, since $f_j(t) = \lambda'_j\big(1 - \cos(t+\nu_j)\big)^{1/2} = \lambda'_j 2^{1/2}|\sin\frac{1}{2}(t+\nu_j)|$. This contradiction shows that $A_k > 1$ for all $k = 1,\ldots,n$.

To complete the proof fix $j, 1 \leq j \leq n$, and choose a point w such that $\varphi_j(w) = 0$. Since $A_j > 1$, w is not real, and by (2.7), $\varphi_k(w) \neq 0$ for all $k \neq j$. Choose a real number t_0. Since $A_k > 1$ for all $k = 1,\ldots,n$ it follows that $\varphi_k(t_0) \neq 0$ for all $k = 1,\ldots,n$. We can, therefore, find a simply connected neighborhood U of t_0 such that $\varphi_k(z) \neq 0$ for all $z \in U$ and all $k = 1,\ldots,n$.

Next, choose a path γ in $\mathbb{C} \setminus \bigcup_{k=1}^n Z_k$ which starts and ends at t_0, winds once around the point w and does not wind around any other point of $\bigcup_{k=1}^n Z_k$. Now, for $k = 1, \ldots, n$, let $F_k(z)$ be an analytic branch of $\lambda_k' \varphi_k(z)^{1/2}$ such that $F_k(t) = f_k(t)$ for all real numbers t in U (see 2.4) and (2.5)). Such branches exist since $\varphi_k(z) \neq 0$ for all $z \in U$ and $k = 1, \ldots, n$, and since U is simply connected. Finally, set $F_0(z) = \sum_{k=1}^n F_k(z)$. Then $F_0(z)$ is analytic in U, and $F_0(t) = \sum_{k=1}^n f_k(t)$ for all real numbers t in U. Therefore, by (2.3),

$$F_0(z) \equiv c \quad \text{in } U. \tag{2.8}$$

For $k = 0, 1, \ldots, n$, let $G_k(z)$ denote the analytic function in U which is obtained by continuing $F_k(z)$ along γ.

The point w is a simple zero of $\varphi_j(z)$, and γ winds once around w and does not wind around any other zero of $\varphi_j(z)$. Therefore $G_j(z) = -F_j(z)$. For any other $k, 0 < k \neq j, \gamma$ does not wind around any zero of $\varphi_k(z)$. Hence $G_k(z) = F_k(z)$ for all $0 < k \neq j$. In view of (2.8), $G_0(z) \equiv c$ in U, and by the Permanence Theorem, $G_0(z) = \sum_{k=1}^n G_k(z)$. Hence $F_j(z) = \frac{1}{2}(F_0(z) - G_0(z)) \equiv 0$ in U. Then $f_j(t) = 0$ for all real numbers t in U, and thus $\lambda_j = 0$, contradicting assumption (iv) of the proposition. This completes the proof. ∎

3. Carathéodory balls and norm balls

THEOREM: *Let $n \geq 2$. The only balls in $H = \{z \in \mathbb{C}^n : \|z\|_1 < 1\}$ in the Carathéodory distance on H, which are balls in the ℓ_1 norm in $\mathbb{C}^n$, are those which are centered at the origin.*

In proving the theorem use will be made of the proposition and the following two lemmas which are quoted from [3, Lemma 1] and [4, Lemma 2.2], cf. [1], [2] and [5]. In the sequel $\|a\|$ will stand for the ℓ_1 norm $\|a\|_1$, of $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$.

LEMMA 1: *Let $\zeta \in \mathbb{C}$, and let $a = (a_1, \ldots, a_n) \in H$ be such that $(a_1, \ldots, a_{j-1}, \zeta, a_{j+1}, \ldots, a_n) \in H$. Then*

$$C\big((a_1, \ldots a_{j-1}, \zeta, a_{j+1}, \ldots a_n), (a_1, \ldots, a_n)\big) = \rho(u_j^{-1}\zeta, u_j^{-1}a_j), \tag{3.1}$$

where

$$u_j = 1 - \sum_{\substack{k=1 \\ k \neq j}}^{n} |a_k|. \tag{3.2}$$

LEMMA 2: *Let $a \in H$ and $\zeta \in \mathbb{C}$ such that $\zeta a \in H$. Then*

$$C(\zeta a, a) = \rho(\|a\|, \|a\|\zeta). \tag{3.3}$$

Proof of the Theorem: For $a \in H$ let

$$B_c(a,r) = \{z \in H : C(z,a) < r\}, \quad \text{and} \quad B_N(a,r) = \{z \in H : \|z - a\| < r\}$$

denote the Carathéodory and the ℓ_1 norm balls, respectively, of radius r centered at a. Suppose that contrary to the statement of the theorem there are points $a \in H\backslash\{0\}$ and $a^N \in H$, and real numbers $0 < \alpha < 1$ and $r_N > 0$ such that

$$B_N\left(a^N, r_N\right) = B_C(a,r) \subset H, \tag{3.4}$$

where

$$r = \tanh^{-1}\alpha. \tag{3.5}$$

Then $\partial B_N(a^N, r_n) = \partial B_C(a,r) \subset H$, where the inclusion follows from the fact that H is bounded and convex, cf. [1, p. 88].

We will show that this assumption leads to a contradiction by considering certain one dimensional subsets of $\partial B_C(a,r)$, which correspond to the following subsets of $\mathbb{C}$:

$$A_j = \{\zeta \in \mathbb{C} : (a_1, \ldots, a_{j-1}, \zeta, a_{j+1}, \ldots, a_n) \in \partial B_C(a,r)\}, \quad j = 1, \ldots n, \tag{3.6}$$

and

$$B = \{\zeta \in \mathbb{C} : \zeta a \in \partial B_C(a,r)\}, \tag{3.7}$$

where r is given in (3.5).

First note that for $\zeta \in A_j, \quad j = 1, \ldots, n,$

$$C\big((a_1, \ldots, a_{j-1}, \zeta, a_{j+1}, \ldots, a_n), (a_1, \ldots, a_n)\big) = r.$$

This and Lemma 1 imply

$$\rho\left(u_j^{-1}\zeta, u_j^{-1}a_j\right) = r = \tanh^{-1}\alpha, \quad u_j = 1 - \sum_{\substack{k=1 \\ k \neq j}}^{n} |a_k|.$$

Therefore, all points $u_j^{-1}\zeta$, $\zeta \in A_j$, lie on an hyperbolic circle in Δ, hyperbolically centered at the point $u_j^{-1}a_j$, which by (1.1) is also a Euclidean circle, whose center and radius can be computed with the aid of (1.2). Hence A_j is a Euclidean circle too which is given by

$$A_j = \left\{ \zeta = \frac{(1-\alpha^2)u_j^2}{u_j^2 - \alpha^2|a_j|^2} \cdot a_j + \alpha u_j \frac{u_j^2 - |a_j|^2}{u_j^2 - \alpha^2|a_j|^2} e^{i\varphi} : 0 \le \varphi \le 2\pi \right\} \tag{3.8}$$

where u_j is given in (3.2).

Suppose, as above, that $\partial B_C(a,r) = \partial B_N(a^N, r_N)$, for some $a^N \in H$ and $r_N > 0$. Fix $j, j = 1, \ldots, n$. Then for all $\zeta \in A_j$

$$r_N = \|(a_1, \ldots, a_{j-1}, \zeta, a_{j+1}, \ldots, a_n) - a^N\| = |\zeta - a_j^N| + \sum_{\substack{k=1 \\ k \ne j}}^{n} |a_k - a_k^N|. \tag{3.9}$$

Since $\zeta \in A_j$ and A_j is a circle, and since all r_N, a_k and a_k^N are constants, it follows that a_j^N must coincide with the center of A_j. Therefore, by (3.8),

$$a_j^N = \frac{(1-\alpha^2)u_j^2}{u_j^2 - \alpha^2|a_j|^2} \cdot a_j, \quad j = 1, \ldots, n \tag{3.10}$$

where u_j is given by (3.2). As a corollary, we get

$$a_j = 0 \quad \text{if and only if } a_j^N = 0. \tag{3.11}$$

Next consider the set B of (3.7). Then, by Lemma 2 and (3.5), $\zeta \in B$ if and only if

$$\tanh^{-1}\alpha = C(\zeta a, a) = \rho(\|a\|\zeta, \|a\|).$$

Consequently, the points $\|a\|\zeta$, $\zeta \in B$, lie on an hyperbolic circle in Δ, hyperbolically centered at the point $\|a\|$. By (1.1) this is also a Euclidean circle. Hence B is a circle in Δ which, in view of (1.2), is given by

$$B = \{\zeta = \lambda + Re^{i\varphi} : 0 \le \varphi \le 2\pi\}, \tag{3.12}$$

where

$$\lambda = \frac{1-\alpha^2}{1-\alpha^2\|a\|^2} \quad \text{and} \quad R = \frac{\alpha}{\|a\|} \cdot \frac{1-\|a\|^2}{1-\alpha^2\|a\|^2}. \tag{3.13}$$

Hence for all $\zeta \in B$,

$$\zeta a = (\lambda + Re^{i\varphi})a \quad \text{and} \quad \zeta a_k = \lambda a_k + R_k e^{i(\varphi+\psi_k)}, \quad 0 \le \varphi \le 2\pi \tag{3.14}$$

where

$$R_k = \alpha \frac{|a_k|(1 - \|a\|^2)}{\|a\|(1 - \alpha^2\|a\|^2)} \quad \text{and} \quad \psi_k = \arg a_k, \quad k = 1, \ldots, n. \tag{3.15}$$

Suppose again that $\partial B_C(a, r) = \partial B_N(a^N, r_N)$. Then for $\zeta \in B$, $\zeta a \in \partial B_C(a, r)$, and by (3.12), (3.13), (3.14) and (3.15)

$$r_N = \|\zeta a - a^N\| = \sum_{k=1}^{n} \left| \lambda a_k + R_k e^{i(\varphi+\psi_k)} - a_k^N \right|, \quad 0 \le \varphi \le 2\pi.$$

Then by the proposition,

$$a_k^N = \lambda a_k, \quad k = 1, \ldots, n, \quad \text{and} \quad a^N = \lambda a, \tag{3.16}$$

where λ is given by (3.13).

We now consider two cases:

CASE 1: $a_j \neq 0$ and $a_k \neq 0$ for some $1 \le j < k \le n$.

CASE 2: $a_j \neq 0$ for some $1 \le j \le n$ and $a_k = 0$ for any other $k \neq j$.

Suppose that we are in Case 1. With no loss of generality we may assume $a_1 \neq 0$, and that $a_k \neq 0$ for some $2 \le k \le n$. Using (3.10), (3.16) and (3.13) for a_1^N we get

$$\frac{(1 - \alpha^2)}{u_1^2 - \alpha^2|a_1|^2} \cdot u_1^2 a_1 = a_1^N = \frac{1 - \alpha^2}{1 - \alpha^2\|a\|^2} \cdot a_1 \tag{3.17}$$

where $u_1 = 1 - \sum_{k=2}^{n} |a_k|$ is as in (3.2). Since $a_1 \neq 0$ and $0 < \alpha < 1$, (3.17) gives

$$u_1^2(1 - \alpha^2\|a\|^2) = u_1^2 - \alpha^2|a_1|^2.$$

By using again the fact that $\alpha \neq 0$ we get $u_1\|a\| = |a_1|$, which is equivalent to

$$\left(1 - \sum_{k=2}^{n} |a_k|\right)\left(|a_1| + \sum_{k=2}^{n} |a_k|\right) - |a_1| = 0$$

or to

$$\left(\sum_{k=2}^{n}|a_k|\right)\left(1-\sum_{k=1}^{n}|a_k|\right)=0.$$

$\|a\|<1$ and $a_k \neq 0$ for some $2 \leq k \leq n$ imply that each factor $\neq 0$, thus leading to a contradiction.

Suppose now that we are in Case 2. With no loss of generality we may assume that $a_2 \neq 0$ and that $a_k = 0$ for all other $k \neq 2$. Then $a = (0, a_2, 0, \ldots, 0)$. By (3.2) $u_2 = 1$, and by (3.10)

$$a_2^N = \frac{1-\alpha^2}{1-\alpha^2|a_2|^2}\cdot a_2 \quad \text{and} \quad a_k^N = 0 \quad \text{for all } k \neq 2. \tag{3.18}$$

Consider the set A_1 of (3.6) with $j = 1$ for $a = (0, a_2, 0, \ldots, 0)$. Then, by (3.2), $u_1 = 1 - |a_2|$, and by (3.8), $\zeta \in A_1$, and $(\zeta, a_2, 0, \ldots, 0) \in \partial B_C(a, r)$, if and only if

$$\zeta = \alpha(1-|a_2|)e^{i\varphi}, \quad 0 \leq \varphi \leq 2\pi. \tag{3.19}$$

Assuming $\partial B_C(a,r) = \partial B_N(a^N, r_N)$, it follows that

$$r_N = \|a^N - (\zeta, a_2, 0, \ldots, 0)\| = |\zeta| + |a_2^N - a_2|$$

and by (3.19) and (3.18) we get

$$r_N = \alpha(1-|a_2|) + \alpha^2|a_2|\frac{1-|a_2|^2}{1-\alpha^2|a_2|^2}. \tag{3.20}$$

We now compute r_N by considering the set B of (3.7). If $\zeta \in B$, then $\zeta a \in \partial B_C(a,r) = \partial B_N(a^N, r_N)$. Here $a = (0, a_2, 0, \ldots, 0), \zeta a = (0, \zeta a_2, 0, \ldots, 0)$ and by (3.10) and (3.11), $a^N = (0, a_2^N, 0, \ldots, 0)$. Therefore, in view of (3.14),

$$r_N = \|\zeta a - a^N\| = |\zeta a_2 - a_2^N| = \left|\lambda a_2 + R_2 e^{i(\varphi+\psi_2)} - a_2^N\right|, \quad 0 \leq \varphi \leq 2\pi.$$

Hence, $\lambda a_2 - a_2^N = 0$, and, consequently, $r_N = |R_2|$. Then, by (3.15)

$$r_N = \alpha\frac{1-|a_2|^2}{1-\alpha^2|a_2|^2}. \tag{3.21}$$

Now, by subtracting the expression for r_N in (3.21) from the expression for r_N in (3.20), we get

$$\frac{\alpha(1-\alpha)|a_2|(1-|a_2|)}{1+\alpha|a_2|} = 0,$$

which is impossible since all factors are positive as $0 < \alpha < 1$ and $0 < |a_2| < 1$. This completes the proof. ∎

Remark: After this paper was accepted for publication the editors informed the author of an alternative proof of the main theorem given by W. Zwonek. Zwonek's paper appears in this volume.

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References

[1] S. Dineen, *The Schwarz Lemma*, Oxford Mathematical Monograph. Clarendon Press, 1989.

[2] T. Franzoni and E. Vesentini, *Holomorphic Maps and Invariant Distances*, North-Holland, Amsterdam, 1980.

[3] B. Schwarz, *Bounds for the Carathéodory distance of the unit ball*, Linear and Multilinear Algebra **32** (1992), 93–101.

[4] B. Schwarz, *Carathéodory balls and norm balls of the domain* $H = \{z_1, z_2) \in \mathbb{C}^2 : |z_1| + |z_2| < 1\}$, Israel Journal of Mathematics **84** (1993), 119–128.

[5] E. Vesentini, *Complex geodesics*, Compositio Mathematica **44** (1981), 375–394.